On the Two Species Asymmetric Exclusion Process with Semi-Permeable Boundaries

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Abstract We investigate the structure of the nonequilibrium stationary state (NESS) of a system of first and second class particles, as well as vacancies (holes), on L sites of a one-dimensional lattice in contact with first class particle reservoirs at the boundary sites; these particles can enter at site 1, when it is vacant, with rate α, and exit from site L with rate β. Second class particles can neither enter nor leave the system, so the boundaries are *semi-permeable*. The internal dynamics are described by the usual totally asymmetric exclusion process (TASEP) with second class particles. An exact solution of the NESS was found by Arita. Here we describe two consequences of the fact that the flux of second class particles is zero. First, there exist (pinned and unpinned) fat shocks which determine the general structure of the phase diagram and of the local measures; the latter describe the microscopic structure of the system at different macroscopic points (in the limit $L \to \infty$) in terms of superpositions of extremal measures of the infinite system. Second, the distribution of second class particles is given by an equilibrium ensemble in fixed volume, or equivalently but more simply by a pressure ensemble, in which the pair potential between neighboring particles grows logarithmically with distance. We also point out an unexpected feature in the microscopic structure of the NESS for finite L: if there are n second class particles in the system then the distribution of first class particles (respectively holes) on the first (respectively last) n sites is exchangeable.

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1 Introduction

In recent work Arita [1, 2], using a matrix ansatz, found the nonequilibrium stationary state (NESS) of a new version of the widely studied one-dimensional totally asymmetric exclusion process (TASEP) [3–11] (see in particular [12] for a recent review of matrix methods for the TASEP). Some other models of two species exclusion processes which can be solved by the matrix ansatz are the partially asymmetric version of the same model, studied in [13], and, in a special case, the model of positive, negative and neutral charges moving under the influence of an electric field studied first in [14]. Common to all these models is the fact that the boundary rates have to be restricted to some special values in order to make the matrix ansatz work.

The model is defined on a subset of the one dimensional lattice $\mathbb{Z}$ consisting of L sites. Each site i, $i = 1, \ldots, L$, may be occupied by a first class particle, occupied by a second class particle, or vacant; vacant sites are also referred to as holes, and first class particles simply as particles. We shall let these three possible states correspond to the values 1, 2, and 0, respectively, of a random variable τ_i; we also introduce the indicator random variables $\eta_a(i)$, $a = 0, 1, 2$, such that $\eta_a(i) = 1$ if $\tau_i = a$ and $\eta_a(i) = 0$ otherwise.

The internal (bulk) dynamics of the system are given by the usual rules for the TASEP with second class particles [9]. The occupation variable τ_i at site i, $i = 1, \ldots, L-1$, attempts when $\tau_i = 1$ or 2 to exchange at rate 1 with τ_{i+1}; when $\tau_i = 1$ the exchange succeeds iff $\tau_{i+1} = 0$ or 2, while for $\tau_i = 2$ it only succeeds if $\tau_{i+1} = 0$. In other words, a first class particle at site i jumps to the right by exchanging with either a hole or second class particle at site $i + 1$, while a second class particle can only jump if the site on its right is empty. At site $i = 1$, first class particles enter the system at rate α provided that site 1 is vacant ($\tau_1 = 0$); at site $i = L$, first class particles leave the system at rate β provided that site L is occupied by a first class particle ($\tau_L = 1$). Second class particles are thus trapped inside the system; since only first class particles can cross the boundaries, we refer to these as *semi-permeable*. (A similar system, but with a different form of semipermeable boundary, was considered in [15]). An equivalent system is obtained by interchanging first class particles with holes, left with right, and α with β, and this symmetry will be reflected in the structure of the NESS. The latter will be determined by the parameters α, β and the density $\gamma = n/L$ of second class particles, where n is the number of second class particles in the system.

The phase diagram of this system in the limit $L \to \infty$ is given in Fig. 1 [2]. The diagram is determined by the distinct formulas for the (first class) particle current J_1 in the different

Fig. 1 The cross section of the phase diagram at a fixed γ
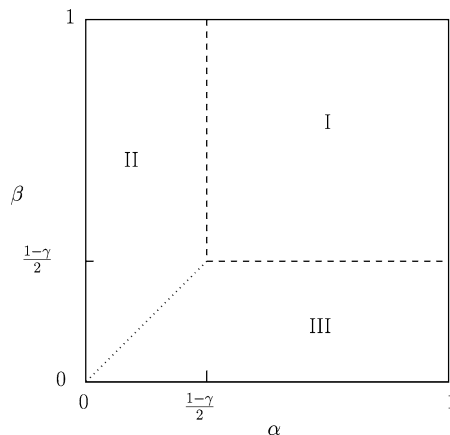
regions:

$$J_1 = \begin{cases} \frac{1-\gamma^2}{4}, & \text{for } \alpha, \beta \geq \alpha_c \text{ (region I)}, \\ \alpha(1-\alpha), & \text{for } \alpha < \alpha_c, \alpha < \beta \text{ (region II)}, \\ \beta(1-\beta), & \text{for } \beta < \alpha_c, \beta < \alpha \text{ (region III)}; \end{cases} \tag{1.1}$$

here the critical value α_c of α and β is

$$\alpha_c = \frac{1-\gamma}{2}. \tag{1.2}$$

The current J_2 of the (trapped) second class particles must vanish and the current J_0 of holes satisfies $J_0 = -J_1$. Note that the form of J_1 as a function of the parameters characterizes the phase plane regions, and that J_1 is continuous but not smooth across all the boundaries.

The phase diagram is similar to that of the open one component TASEP [7, 8] and indeed in the limit $\gamma \to 0$ reduces to it; moreover, in regions II and III the current is independent of γ and takes the same values as in the one component case, although the size of these regions shrinks as γ increases. As we discuss in Sect. 6, however, there will be residual differences between the local microscopic states of the one species model and $\gamma \to 0$ limit of the two species model; in particular, there remain an infinite number of second class particles near one or both boundaries of the two species system. Note also that there is a discontinuity, equal to γ, in the derivative of J_1 with respect to α (β) on the I/II (I/III) boundary; one might say that the order of the phase transition in J_1 when $\gamma \neq 0$ differs from that when $\gamma = 0$.

The macroscopic density profiles $\rho_a(x)$ in the NESS, $a = 0, 1, 2$, defined by

$$\rho_a(x) = \lim_{L \to \infty, n/L \to \gamma, i/L \to x} \langle \eta_a(i) \rangle, \quad 0 \leq x \leq 1, \tag{1.3}$$

with $\langle \cdot \rangle$ the expected value in the NESS, have been computed in [2]; the results are summarized in Table 1 (but see Remark 1.1 below). Knowing any two of these densities determines the third, via $\sum_a \rho_a(x) = 1$. In fact, knowing $\rho_1(x)$ for all x and all values of α and β determines $\rho_0(x)$, from the particle-hole symmetry, and hence all profiles, but for clarity we give in Table 1 both $\rho_1(x)$ and $\rho_0(x)$. In region II the system divides itself into two parts, $x < x_0$ and $x > x_0$, with different formulas for $\rho_0(x)$; similarly in region III there are

Table 1 Density profiles in different regions of the phase plane. Note that x_0 is defined only in region II and on its boundaries, and x_1 only in region III and on its boundaries

Region	$\rho_1(x)$			$\rho_0(x)$		
I	α_c			α_c		
I/II boundary	α_c			α_c		
I/III boundary	α_c			α_c		
	$x < x_1$		$x > x_1$	$x < x_0$		$x > x_0$
II		α		$1-\alpha$		α
III	β		$1-\beta$		β	
II/III boundary (Shock Line)	$\alpha(=\beta)$		linear	linear		$\alpha(=\beta)$

different formulas for $\rho_1(x)$ for $x < x_1$ and $x > x_1$. Here

$$x_0 = 1 - \frac{\gamma}{1 - 2\alpha}, \quad \alpha \le \beta, \ \alpha < \alpha_c;$$

$$x_1 = \frac{\gamma}{1 - 2\beta}, \quad \beta \le \alpha, \ \beta < \alpha_c.$$

(1.4)

On the II/III boundary $\alpha = \beta < \alpha_c$, the *shock line*, the profiles include linear regions:

$$\rho_0(x) = \begin{cases} \frac{x_0 - x}{x_0}(1 - \alpha) + \frac{x}{x_0}\alpha, & 0 \le x \le x_0, \\ \alpha, & x_0 \le x \le 1 \end{cases}$$

$$\rho_1(x) = \begin{cases} \alpha, & 0 \le x \le x_1, \\ \frac{1-x}{1-x_1}\alpha + \frac{x-x_1}{1-x_1}(1 - \alpha), & x_1 \le x \le 1. \end{cases}$$

(1.5)

These arise from averaging over the position of a shock, as in the one species TASEP; further discussion is given below.

Remark 1.1

(a) The density values $\rho_a(x)$ at the boundaries $x = 0, 1$ and at the fixed shocks $x = x_0, x_1$ may depend the way the limit (1.3) is taken. The boundary cases $x = 0, 1$ were discussed in [2] except on the I/II and I/III boundaries. We discuss the limits at x_0, x_1 in Sects. 5 and 6; this gives some further information about limits at the boundaries since $x_0 = 0$ and $x_1 = 1$ on the I/II and I/III boundaries, respectively.

(b) In the one-component model the phase plane regions corresponding to I, II, and III are called the maximum current, low density, and high density regions, respectively. We do not adopt that terminology here since in region III the particle density is low for $x < x_1$.

Some typical profiles, obtained from simulations, are shown in Fig. 2. Since these are from a finite system, they do not coincide perfectly with the description in Table 1: there are boundary effects, and the density transitions in regions II and III, at x_1 and x_0 respectively, have nonzero width of order $\sqrt{L}$. This is related to the nonuniqueness of the limit (1.3) at these points, as mentioned above, and is discussed in Sect. 5.

We now give an intuitive discussion of some of the phenomena that give rise to these profiles. Consider a macroscopically uniform portion of our system in the limit $L \to \infty$, with densities of holes, particles and second class particles denoted by ρ_0, ρ_1 and ρ_2, respectively, where $\rho_0 + \rho_1 + \rho_2 = 1$. In such a region the measure will be the known translation invariant measure, with these densities, for the two species TASEP (see [9, 16, 17] and the discussion in Sect. 8); in this measure (which is not a product measure) the first class particles considered separately, and the holes considered separately, are distributed according to product measures, so that $J_1 = \rho_1(1 - \rho_1)$ and $J_0 = -\rho_0(1 - \rho_0)$. Thus from $J_0 = -J_1$ it follows that in any uniform stretch of the NESS either

$$\rho_1 = \rho_0 = (1 - \rho_2)/2 \quad \text{or} \quad \rho_2 = 0, \quad \rho_1 = 1 - \rho_0.$$

(1.6)

This fact, which may be seen in the results of [2], is key to understanding the gross structure of the densities in different regions of the phase diagram.

These density profiles differ from those the single-species open TASEP in two notable ways: in regions II and III the density profiles have a point of discontinuity, and on the II/III
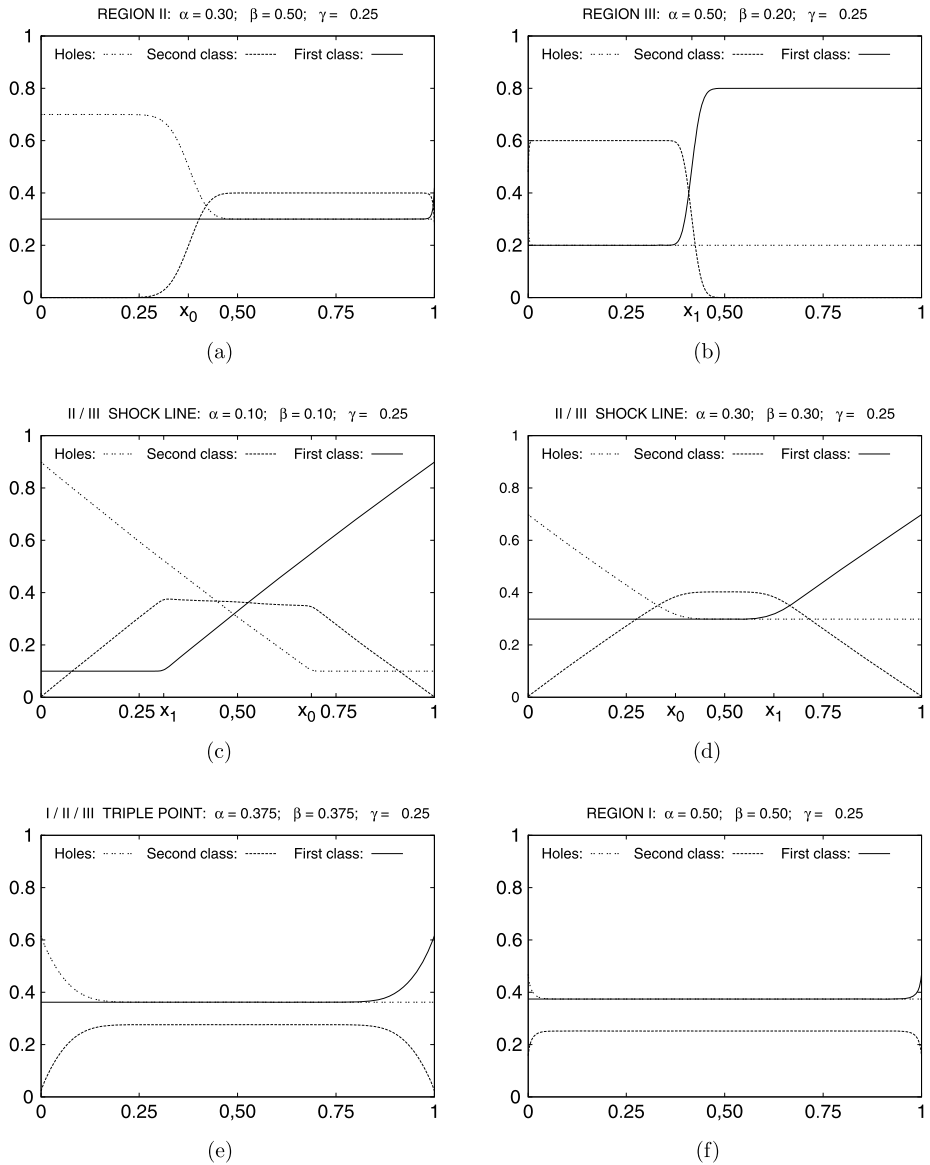
Fig. 2 Density profiles in a system with $L = 1000$

boundary the linear region occupies only part of the system. These and other properties can be understood in terms of the occurrence of a *fat shock*. By this term we refer, not to a broadening of the sharp shock usually seen in the TASEP (as can occur in the partially asymmetric model [18]), but rather to a macroscopically uniform interval which contains all the second-class particles, and thus conforms to the first alternative in (1.6). We may think of this fat shock as bounded by shocks of the usual sort occurring in two different one species TASEP systems which are obtained by making appropriate identifications of second class particles with either first class particles or holes. To see how this occurs, recall that if one either

Fig. 3 Shock interpretation at $\alpha = \beta < \alpha_c$. Densities $\rho_a(x)$ are plotted against x, with the convention that the height at x of the region labeled with particle type a is $\rho_a(x)$. For example, the density of second class particles in the shock region is $(1-\alpha) - \alpha = 1 - 2\alpha$ and the density of holes to the left of the shock is $1 - \alpha$. The fat shock may in fact be located anywhere in the system
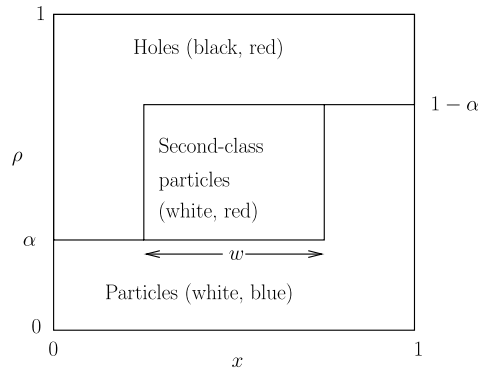
identifies first and second class particles by coloring holes black and both kinds of particles white, or else identifies second-class particles and holes by coloring these species red and first-class particles blue, then the black/white particles, as well as the red/blue particles, form standard two species TASEPs in the bulk. The dynamics at the boundaries is different, since some white "particles" or red "holes" will be trapped in the system. A careful justification of the conclusions below is given in Sects. 4 and 5.

Consider now the behavior of the system on the boundary of regions II and III (the shock line). Then by previous analysis, see e.g. [10], one knows that a typical profile for the one species model contains a shock between a region of density α on the left and $1 - \alpha$ on the right; the shock position has mean velocity zero and its (fluctuating) position is uniformly distributed over the system. We see this same behavior for both the black/white and red/blue systems described above, with the black/white shock necessarily located to the left of the red/blue one. The *typical* profile at any given time looks on the macroscopic scale like Fig. 3, where the convention is that at any point x the height of the region labeled with particle type a is $\rho_a(x)$.

Clearly both shock fronts have mean velocity zero and are trapped in the system, and since the total number of second-class particles is γL, the macroscopic width w of the fat shock must satisfy $w = \gamma/(1 - 2\alpha)$. This forces the two shock fronts to move (i.e., fluctuate) in collusion so as to keep the macroscopic shock width fixed; we expect this fluctuation, as for the shock in the single component TASEP on its shock line, to be on a diffusive time scale growing as L^2. The density profiles $\rho_a(x)$ arise as averages over the shock position, and this gives rise to the linear profiles (1.5); in contrast to the situation in the one species case, however, here they occupy only part of the system because the shock can fluctuate only over an interval of width $1 - w$. The shock fluctuation is also reflected in the structure of the local measures obtained in the limit $L \to \infty$, which are superpositions of states with different densities (see Sect. 5). The critical value of α occurs when the fat shock fills the system, i.e., when $w = 1$, from which we regain (1.2).

The situation in regions II and III is similar. The fat shock width is in general

$$w(\alpha, \beta, \gamma) = \frac{\gamma}{1 - 2\alpha \wedge \beta}, \tag{1.7}$$

where $\alpha \wedge \beta = \min\{\alpha, \beta\}$; in region II the shock is pinned to the right boundary, and in region III is pinned to the left boundary. Since the shock is fixed it gives rise to discontinuities in the density profiles; see Fig. 2 as well as the discussion of a related model, where similar behavior occurs, in Sect. 8 (Fig. 4). There is no corresponding discontinuity in the single-species TASEP ($\gamma = 0$) because in that case there is a single shock of zero macroscopic

width. In region I, w as given in (1.7) is greater than 1 and the fat shock fills the system; the density profiles are uniform and conform to the first alternative of (1.6).

The outline of the rest of the paper is as follows. In Sect. 2 we discuss the matrix method for this system. We use a different representation of the matrices from that of [2], which makes it easier to prove certain features of the NESS discussed later. In Sect. 3 we show that the marginal distribution induced by the NESS on particles in the first n states of the system, and on holes in the last n, is exchangeable, i.e., that the probability of finding r first class particles (holes) on some specified set of r sites among the first (last) n is independent of the choice of sites. In Sect. 4 we establish the fat shock picture described above. In Sect. 5 determine the local measures, in the bulk, for the infinite volume limit of the system, and in Sect. 6 consider the local measures near the boundaries, focusing on a Bernoulli property which is a consequence of the exchangeability established in Sect. 3. In Sect. 7 we show that the second class particles form an equilibrium system, most simply described by a pressure ensemble. This is related, in our case, to the fact that the current of second class particles is zero. For similar situations, see [19, 20].

In Sect. 8 we make some concluding remarks and, in particular, describe some closely related models. One such model is a generalization of the standard "defect particle" model; another describes a system of first class particles, second class particles, and holes on a ring with one semi-permeable bond which second class particles cannot cross. Several more technical remarks are recorded in the appendices.

2 The Matrix Ansatz

The stochastic system described in Sect. 1 is ergodic in finite volume L and thus there exists a unique invariant measure $\mu_{L,n}^{\alpha,\beta}$ on the configuration space

$$Y_{L,n} \equiv \{(\tau_1, \ldots, \tau_L) \mid \tau_i = 0, 1, 2; \ \tau_i = 2 \text{ for } n \text{ values of } i\}, \tag{2.1}$$

where from now on we will assume that $0 < n < L$. This measure may be obtained from a matrix ansatz [2], combining the matrix algebra of [9] (which discussed the system with the same constituents as in the current work, but on a ring) with the treatment of the one species open system via matrix-elements from [7]. One introduces matrices X_0, X_1, and X_2 and vectors $|V_\beta\rangle$ and $\langle W_\alpha|$ which satisfy

$$X_1 X_0 = X_1 + X_0, \qquad X_1 X_2 = X_2, \qquad X_2 X_0 = X_2, \tag{2.2}$$

$$\langle W_\alpha | X_0 = \frac{1}{\alpha} \langle W_\alpha|, \qquad X_1 | V_\beta \rangle = \frac{1}{\beta} | V_\beta \rangle. \tag{2.3}$$

Then for a configuration $\tau = (\tau_1, \ldots, \tau_L) \in Y_{L,n}$ the probability of τ in the invariant measure is

$$\langle \tau \rangle_{\mu_{L,n}^{\alpha,\beta}} = Z^{\alpha,\beta}(L, n)^{-1} \langle W_\alpha | X_{\tau_1} \cdots X_{\tau_L} | V_\beta \rangle, \tag{2.4}$$

where $Z^{\alpha,\beta}(L, n)$ is the normalization factor

$$Z^{\alpha,\beta}(L, n) = \sum_{\tau \in Y_{L,n}} \langle W_\alpha | X_{\tau_1} \cdots X_{\tau_L} | V_\beta \rangle, \tag{2.5}$$

which, with a slight misuse of the nomenclature of equilibrium statistical mechanics, we call the *partition function*. We will frequently omit superscripts such as α, β in (2.5) when no confusion can arise.

We will work throughout in a realization of (2.2)–(2.3) for which the matrices and vectors have the properties

$$X_2 = X_1 X_0 - X_0 X_1 = |V_1\rangle\langle W_1|, \qquad X_2|V_\beta\rangle = |V_1\rangle, \qquad \langle W_\alpha|X_2 = \langle W_1|, \quad (2.6)$$

$$\langle W_\alpha|V_1\rangle = \langle W_1|V_\beta\rangle = 1 \quad \text{for all } \alpha, \beta, \tag{2.7}$$

where $|V_1\rangle$ and $\langle W_1|$ are vectors satisfying (2.3). Note that X_2 is then a one-dimensional projection operator. The realization is given in Appendix A, but we will need no consequences beyond (2.6) and (2.7). Because of (2.7) we make the convention that $Z^{\alpha,1}(0,0) = Z^{1,\beta}(0,0) = 1$.

Remark 2.1 The nature of X_2 in this representation shows that certain distributions obtained from (2.4) factorize. Let $Q_1, \ldots, Q_n$ denote the (random) positions of the second class particles in the system (note that these can be ordered once and for all). Then the probability that the j_1^{th}, j_2^{th}, ..., j_m^{th} second class particles are located on sites $q_{j_1}, \ldots, q_{j_m}$ is

$$\mu_{L,n}^{\alpha,\beta}(Q_{j_i} = q_{j_i}, i = 1, \ldots, m)$$
$$= Z^{\alpha,\beta}(L,n)^{-1} Z^{\alpha,1}(q_{j_1} - 1, j_1 - 1)$$
$$\times \prod_{i=2}^{m} Z^{1,1}(q_{j_i} - q_{j_{i-1}} - 1, j_i - j_{i-1} - 1) Z^{1,\beta}(L - q_{j_m}, n - j_m). \tag{2.8}$$

Moreover, if we condition on the event that $Q_j = q$, i.e., that the jth second class particle is located at site q, then the conditional measure is a product of the measures associated with the sites before and after j, so that if τ is a configuration consistent with this event then

$$\mu_{L,n}^{\alpha,\beta}(\tau_1, \ldots, \tau_L \mid Q_j = q) = \mu_{q-1,j-1}^{\alpha,1}(\tau_1, \ldots, \tau_{q-1}) \mu_{L-q,n-j}^{1,\beta}(\tau_{q+1}, \ldots, \tau_L). \tag{2.9}$$

A factorization property of this type is also known for the translation invariant measures for the two species TASEP [9, 17]. Similar expressions are easily obtained when conditioning on the presence of several specified second class particles at specified sites. We will use (2.8) and (2.9) in Sects. 4 and 5, when we discuss the fat shock and describe local measures in the NESS.

3 Exchangeability of Measures

In this section we demonstrate a remarkable property of the finite-volume NESS with n second class particles: the *exchangeability* [21] of the measure on first class particles within the first n sites, or equivalently on holes in the last n sites. Specifically, this means that for any $r \leq n$ the probability of finding first class particles on the r sites $1 \leq i_1 < i_2 < \ldots < i_r \leq n$ depends only on r, i.e., is independent of the choice of positions $i_1, i_2, \ldots, i_r$. When $r = 1$ this is implicit in (38) of [2], although it is not emphasized. As a consequence of the ideas of the proof we will also obtain, for any i, j with $i, j \geq 1$ and $i + j - 1 \leq L$, the probability of finding a block of j consecutive first class particles starting at site i; this generalizes the density formula of [2], which corresponds to $j = 1$.

The key quantity for our arguments is the probability of finding (first class) particles at sites $i_1, \ldots, i_{r-1}$ together with a block of j particles starting at site i_r, where $r \geq 1$, $i_1 < \cdots < i_{r-1}$, and $i_r > i_{r-1} + 1$; we allow $j = 0$, with the interpretation that in this case there is no restriction on what happens at site i_r or any succeeding sites. Thus the probability in question is $Z(L, n)^{-1} E_r(L, n; i_1, \ldots, i_r; j)$, where E_r is a sum of weights for certain configurations $\tau \in Y_{L,n}$:

$$E_r(L, n; i_1, \ldots, i_r; j) = \sum_{\substack{\tau_{i_1} = \ldots = \tau_{i_{r-1}} = 1 \text{ if } r \geq 2 \\ \tau_{i_r} = \tau_{i_r+1} = \ldots = \tau_{i_r+j-1} = 1 \text{ if } j \geq 1}} \langle W_\alpha | X_{\tau_1} \ldots X_{\tau_L} | V_\beta \rangle. \tag{3.1}$$

In (3.1) we must have $i_r + j - 1 \leq L$, since there are only L sites, and $r + j - 1 \leq L - n$, since there are n second class particles. For certain parts of the analysis we will have to consider separately two cases, in which these two inequalities respectively provide the effective bounds on j:

Case 1: $i_r \geq n + r$, so that $0 \leq j \leq L - i_r + 1$;
Case 2: $i_r \leq n + r - 1$, so that $0 \leq j \leq L - n - r + 1$.

We will analyze the E_r using a simple recursion:

$$E_r(L, n; i_1, \ldots, i_r; j) = E_r(L, n; i_1, \ldots, i_r; j + 1) + E_r(L - 1, n; i_1, \ldots, i_r; j - 1). \tag{3.2}$$

This holds whenever all terms are defined, which requires that j be positive and satisfy $j \leq L - i_r$ in Case 1 and $j \leq L - n - r$ in Case 2. To derive (3.2) we consider the value of τ_{i_r+j} in each term of the sum (3.1). Terms with $\tau_{i+j} = 1$ sum precisely to $E_r(L, n; i_1, \ldots, i_r; j + 1)$, and for terms with $\tau_{i+j} = 0$ or $\tau_{i+j} = 2$ we use the matrix algebra to reduce

$$X_{\tau_{i_r+j-1}} X_{\tau_{i_r+j}} = \begin{cases} X_1 X_0 = X_1 + X_0, & \text{if } \tau_{i_r+j} = 0, \\ X_1 X_2 = X_2, & \text{if } \tau_{i_r+j} = 2; \end{cases} \tag{3.3}$$

the resulting sum is just $E_r(L - 1, n; i_1, \ldots, i_r; j - 1)$.

To determine the E_r the recursion (3.2) must be supplemented by boundary conditions at the maximum and minimum values of j. When $j = 0$, (3.1) gives

$$E_r(L, n; i_1, \ldots, i_r; 0) = \begin{cases} E_{r-1}(L, n; i_1, \ldots, i_{r-1}; 1), & \text{if } r \geq 2, \\ Z(L, n), & \text{if } r = 1. \end{cases} \tag{3.4}$$

The value of E_r for j maximal is case dependent. In Case 1, if j takes its maximum possible value $L - i_r + 1$ then each matrix product in (3.1) ends with $X_1^{L-i_r+1} | V_\beta \rangle = \beta^{-(L-i_r+1)} | V_\beta \rangle$, so that

$$E_r(L, n; i_1, \ldots, i_r; L - i_r + 1)$$
$$= \begin{cases} \beta^{-(L-i_r+1)} E_{r-1}(i_r - 1, n; i_1, \ldots, i_{r-1}; 1), & \text{if } r \geq 2, \\ \beta^{-(L-i_r+1)} Z(i_r - 1, n), & \text{if } r = 1, \end{cases} \quad \text{(Case 1)}. \tag{3.5}$$

In Case 2, if j takes its maximum possible value $L - n - r + 1$ then the rightmost factor in the matrix product in (3.1) is X_2 and there are no factors of X_0; using the matrix algebra relations $X_1 X_2 = X_2$, $X_2^2 = X_2$ and $X_2 | V_\beta \rangle = | V_1 \rangle$, we have that

$$E_r(L, n; i_1, \ldots, i_r; L - n - r + 1) = 1 \quad \text{(Case 2)}. \tag{3.6}$$

Lemma 3.1 *The recursion* (3.2), *together with the boundary conditions* (3.4) *and either* (3.5) *or* (3.6), *determines* $E_r(L, n; i_1, \ldots, i_r; j)$ *by an inductive computation.*

Proof The primary induction is on increasing values of r, with n held fixed throughout. The inductive assumption that the E_{r-1} are known is needed when (3.4) or (3.5) is applied, and since for $r = 1$ the right hand side of these equations is a (known) partition function one may treat all values $r \geq 1$ uniformly. Then for fixed r ($r \geq 1$) and $i_1, \ldots, i_r$ we induce on increasing values of L: for the minimum possible value, $L = i_r$, we must be in Case 1 and have $j = 0$ or $j = 1$, so that all $E_r(L, n; i_1, \ldots, i_r; j)$ are determined by (3.4) and (3.5). For any larger value of L either (3.5) or (3.6) determines $E_r(L, n; i_1, \ldots, i_r; j)$ for the maximum possible value of j and we may then induce downward on j using (3.2).   □

We can now verify exchangeability; we show that $E_r(L, n; i_1, \ldots, i_r; j)$ is equal to the corresponding weight with the sites $i_1, \ldots, i_r$ in standard positions $1, \ldots, r$.

Theorem 3.2 *Let* $1 \leq i_1 < i_2 < \ldots < i_r$ *be sites such that* $i_r \leq n + r - 1$, *and let* j *be an integer less than or equal to* $L - n - r + 1$. *Then*

$$E_r(L, n; i_1, \ldots, i_r; j) = E_1(L, n; 1; r + j - 1). \tag{3.7}$$

Proof The proof is by induction on r. By Lemma 3.1 it suffices to show that $E_1(L, n; 1; r + j - 1)$ satisfies the same recurrence relation (3.2) and boundary conditions (3.4), (3.6) as does $E_r(L, n; i_1, \ldots, i_r; j)$. This follows immediately from the corresponding relations for E_1 and, for $r \geq 2$, the induction assumption.   □

We finally give the explicit formula for E_1 which, by (3.7), also provides a formula for $E_r(L, n; i_1, \ldots, i_r; j)$ when $i_r \leq n + r - 1$. This result will not be needed in the remainder of the paper.

The formula involves the Catalan triangle numbers [22]

$$C_n^m = \binom{m + n}{n} \frac{m - n + 1}{m + 1}, \tag{3.8}$$

which satisfy the recursion

$$C_n^{m-1} + C_{n-1}^m = C_n^m \tag{3.9}$$

and the boundary conditions

$$C_{-1}^m = 0, \qquad C_0^m = 1, \qquad C_m^m = C_{m-1}^m = \frac{1}{m + 1}\binom{2m}{m}. \tag{3.10}$$

We then define the additional constants

$$c_{j,k} = \begin{cases} C_{k-j}^{k-1}, & \text{if } j \geq 1, \\ \delta_{k,0}, & \text{if } j = 0; \end{cases} \tag{3.11}$$

$$d_{m,j,k} = \begin{cases} \binom{m-j+k}{k} - \binom{m-j+k}{k-j}, & \text{if } j \leq m, \\ \delta_{k,0}, & \text{if } j = m + 1. \end{cases} \tag{3.12}$$

In (3.11)–(3.12) the convention is that $\binom{p}{q} = 0$ for integer p, q with $p \geq 0$, $q < 0$.

Theorem 3.3 *Case 1: If $n + 1 \leq i \leq L$ and $0 \leq j \leq L + 1 - i$ then*

$$E_1(L, n; i; j) = \sum_{k=j}^{L-i} c_{j,k} Z(L - k, n) + Z(i - 1, n) \sum_{k=0}^{L-i-1} d_{L-i,j,k} \beta^{-L+i+k-1}. \quad (3.13)$$

Case 2: If $1 \leq i \leq n$ and $0 \leq j \leq L - n$ then

$$E_1(L, n; i; j) = \sum_{k=j}^{L-n} c_{j,k} Z(L - k, n), \quad (3.14)$$

Proof Case 1: We temporarily denote the right hand side of (3.13) by $F(L, n; i; j)$. By Lemma 3.1 it suffices to verify that F satisfies relations corresponding to (3.4), (3.5), and (3.2). It will be convenient to denote the two terms in (3.13) by $F_1(L, n; i; j)$ and $F_2(L, n; i; j)$, respectively.

Since $d_{L-i,0,k} = 0$ and $c_{0,k} = \delta_{k,0}$ we have immediately that $F(L, n; i; 0) = Z(L, n)$ (compare (3.4)). Moreover, the sum defining $F_1(L, n; i; L - i + 1)$ is empty and so from $d_{L-i,L-i+1,k} = \delta_{k,0}$ we have $F(L, n; i; L - i + 1) = \beta^{-L+i-1} Z(i - 1, n)$ (compare (3.5)). It remains to check the equivalent of (3.2), which we shall show is satisfied by F_1 and F_2 separately; recall that in (3.2), $1 \leq j \leq L - i$. For $j \geq 2$,

$$F_1(L, n; i; j + 1) + F_1(L - 1, n; i; j - 1)$$

$$= \sum_{k=j+1}^{L-i} C_{k-j-1}^{k-1} Z(L - k, n) + \sum_{k=j-1}^{L-1-i} C_{k-j+1}^{k-1} Z(L - 1 - k, n)$$

$$= \sum_{k=j}^{L-i} \left(C_{k-j-1}^{k-1} + C_{k-j}^{k-2} \right) Z(L - k, n) = F_1(L, n; i; j), \quad (3.15)$$

where we have used $C_{-1}^{j-1} = 0$ (see (3.10)) and $C_{k-j-1}^{k-1} + C_{k-j}^{k-2} = C_{k-j}^{k-1}$ (see (3.9)). The case $j = 1$ is easily checked separately. Similarly one verifies that

$$F_2(L, n; i; j + 1) + F_2(L - 1, n; i; j - 1) = F_2(L, n; i; j) \quad (3.16)$$

separately for $j \leq L - i - 1$ and for $j = L - i$.

Case 2: We denote the right hand side of (3.14) by $G(L, n; i; j)$, and show that G satisfies the appropriate boundary conditions and recursion. One checks immediately that $G(L, n; i; 0) = Z(L, n)$ (compare (3.4)) and, using (3.10), that $G(L, n; i; L - n) = 1$ (compare (3.6)). Finally one shows that, for $1 \leq j \leq L - n - 1$,

$$G(L, n; i; j + 1) + G(L - 1, n; i; j - 1) = G(L, n; i; j); \quad (3.17)$$

the proof is essentially the same as that of the recursion for F_1 in Case 1. $\qquad \square$

4 The Fat Shock

In this section we give a precise definition and analysis of the fat shock discussed informally in the introduction. The analysis will be used in the next section for the determination of

local states in the infinite volume limit. We define the fat shock microscopically as the region between the positions Q_1 and Q_n of the first and last second class particles in the system.

The joint distribution of Q_1 and Q_n was obtained in Remark 2.1; it is convenient here to write this, for $j, k, l \geq 0$, as

$$
\begin{aligned}
\theta_{L,n}^{\alpha,\beta}(j,k,l) &\equiv \mu_{L,n}^{\alpha,\beta}(Q_1 = j+1, Q_n = j+k+2)\delta_{j+k+l,L-2} \\
&= \frac{Z^{\alpha,1}(j,0)Z^{1,1}(k,n-2)Z^{1,\beta}(l,0)}{Z^{\alpha,\beta}(L,n)}\delta_{j+k+l,L-2}.
\end{aligned}
\tag{4.1}
$$

We can determine the large-L behavior of θ by replacing the partition functions in (4.1) with their asymptotic forms; these can be obtained from [7] and [2], and are summarized in Appendix B. In some cases it is convenient to further approximate the distribution of k, which represents the fat shock width on the microscopic scale, by a Gaussian. (Recall that a macroscopic width $w = w(\alpha, \beta, \gamma)$ for the fat shock was predicted on heuristic grounds in Sect. 1 (see (1.7)), so we expect that $k \sim wL$ for large L.) We omit details of the computations and summarize the results in the next remark, where the notation $\theta_{L,n}^{\alpha,\beta}(j,k,l) \sim f(\alpha, \beta, \gamma, j, k, l)$ indicates that the ratio of the two quantities approaches 1 as $L \to \infty$ with $n = \lfloor \gamma L \rfloor$, where $\lfloor u \rfloor$ is the greatest integer contained in u.

Remark 4.1 (a) On the boundary of regions II and III ($\alpha = \beta < \alpha_c$),

$$
\theta_{L,n}^{\alpha,\alpha}(j,k,l) \sim \frac{1}{L(1-w)}\sqrt{\frac{A(\alpha)}{\pi L}}e^{-A(\alpha)(k-Lw)^2/L}\delta_{j+k+l,L-2},
\tag{4.2}
$$

where $A(\alpha) = (1-2\alpha)^3/(4\gamma\alpha(1-\alpha))$. That is, under $\theta_{L,n}^{\alpha,\alpha}(j,k,l)$, k is approximately Gaussian with mean Lw and variance of order L, j is approximately uniformly distributed on the range $0 \leq j \leq L - l - 2$, and $l = L - 2 - j - k$. On the macroscopic scale, this means that the width of the fat shock is w and its left endpoint is uniformly distributed on the interval $[0, 1-w]$.

(b) In region II ($\alpha < \alpha_c, \alpha < \beta$),

$$
\theta_{L,n}^{\alpha,\beta}(j,k,l) \sim p^{\alpha(1-\alpha),\beta}(l)\sqrt{\frac{A(\alpha)}{\pi L}}e^{-A(\alpha)(k-Lw)^2/L}\delta_{j+k+l,L-2},
\tag{4.3}
$$

where we have introduced the (normalized) probability distribution

$$
p^{u,\beta}(l) = \frac{\beta(1+\sqrt{1-4u})-2u}{2\beta}u^l Z^{\beta,1}(l,0), \quad l = 0, 1, \dots,
\tag{4.4}
$$

defined for $u < \beta(1-\beta)$ if $\beta \leq 1/2$, $u \leq 1/4$ otherwise. $p^{u,\beta}(l)$ decreases exponentially for large l unless $u = 1/4$, when the decrease is as $l^{-3/2}$ (see (B.1)); p is normalized by (B.2). Thus on the microscopic scale l is typically of order 1 and the shock width k is distributed as in (a). On the macroscopic scale the fat shock has width w and is pinned to the right end of the system. The analysis in region III is similar.

(c) On the boundary of regions I and II ($\alpha_c = \alpha < \beta$),

$$
\theta_{L,n}^{\alpha,\beta}(j,k,l) \sim p^{\alpha(1-\alpha),\beta}(l)2\sqrt{\frac{A(\alpha)}{\pi L}}e^{-A(\alpha)(k-L)^2/L}\delta_{j+k+l,L-2}.
\tag{4.5}
$$

This is as in (b) except that here $w = 1$ and as a result k is distributed as a Gaussian random variable conditioned to have value at most equal to its mean, and there is a corresponding factor of 2 in the normalization. The analysis on the I/III boundary is similar.

(d) At the triple point ($\alpha_c = \alpha = \beta$),

$$\theta_{L,n}^{\alpha,\beta}(j,k,l) \sim \frac{2\gamma^2}{L(1-\gamma)^2} e^{-A(\alpha)(k-L)^2/L} \delta_{j+k+l,L-2}. \tag{4.6}$$

The distribution of k is as in (c) but here j and l are free, subject only to the constraint $j + l = L - 2 - k$.

(e) In region I ($\alpha_c < \alpha, \beta$),

$$\theta_{L,n}^{\alpha,\beta}(j,k,l) \sim p^{(1-\gamma^2)/4,\alpha}(j) p^{(1-\gamma^2)/4,\beta}(l) \delta_{j+k+l,L-2}; \tag{4.7}$$

j and l are both of order 1 (microscopically) and k is determined by the delta function constraint.

Note that the results of Remark 4.1 confirm the picture of the fat shock behavior sketched in Sect. 1.

5 Local States in the Infinite Volume Limit in the Bulk

In this section we discuss a question inspired by the treatment of the one component system by Liggett [10]: is there a *local state*, in the infinite volume limit, at the scaled position $x \in [0, 1]$ (that is, near site $i = \lfloor xL \rfloor$, with $L \to \infty$), and if so what is it? Informally, the local state of the system at position x is the state observed by zooming in on that position and letting the size of the system approach infinity.

To formulate a precise question we consider a limit in which n and i increase with L in such a way that $i \to \infty$, $L - i \to \infty$, $i/L \to x \in [0, 1]$, and $n/L \to \gamma \in (0, 1)$. In this setting we ask about the existence and nature of the weak limit $\lim_{L\to\infty} T^{-i} \mu_{L,n}^{\alpha,\beta}$, where T is translation by one lattice site and so the operator T^{-i} carries site i to the origin; equivalently, we consider the sites of our open system to run from $1 - i$ to $L - i$ and look at the probabilities of configurations in the interval from $-K$ to K, take $L, i, L - i$, and n to infinity as above, and then make K arbitrary. The limit (if it exists) is a measure on the configuration space $Y = \{0, 1, 2\}^{\mathbb{Z}}$; we call it a *local state in the bulk* since (in the $L \to \infty$ limit) it describes a situation infinitely far from each boundary; the *local state at the boundary* will be discussed in Sect. 6.

It will suffice to consider a special class of these limiting procedures; specifically, we will always take

$$n = n_L = \lfloor \gamma L \rfloor \quad \text{and} \quad i = i_L = \lfloor xL \rfloor + c\sqrt{L}; \tag{5.1}$$

we must assume that $c > 0$ if $x = 0$ and $c < 0$ if $x = 1$. We then define

$$\mu_{x,c} \equiv \lim_{L\to\infty} T^{-i_L} \mu_{L,n_L}^{\alpha,\beta}. \tag{5.2}$$

The limit in (5.2) certainly exists along subsequences, by the compactness of the set of measures on Y. To simplify notation we will ignore the necessity of passing to subsequences; since the limiting measure will be found to be unique, the limit of the sequence itself must also exist. For most values of the parameters the limit will in fact be independent of the

choice of c, but this is not true when $x = x_0$ in region II or on the I/II boundary, or $x = x_1$ in region III or on the I/III boundary.

We first consider the currents and densities in the state $\mu_{x,c}$. The currents in the finite system, and hence also in the limit, are independent of the site:

$$\langle \eta_1(j)(1 - \eta_1(j+1)) \rangle_{\mu_{x,c}} = \langle (1 - \eta_0(j-1))\eta_0(j) \rangle_{\mu_{x,c}} = J_1 \tag{5.3}$$

for any j, with J_1 given in (1.1). The limiting densities $\rho_a(x, c)$ for $a = 0, 1, 2$ are defined by

$$\rho_a(x, c) = \lim_{L \to \infty} \langle \eta_a(i_L) \rangle_{\mu_{L,n_L}^{\alpha,\beta}} = \langle \eta_a(0) \rangle_{\mu_{x,c}}, \tag{5.4}$$

with the last equality expressing the fact that i_L corresponds to the origin in $\mu_{x,c}$. It is easy to check, from the asymptotic computations of [2], that the limit in (5.4) would be unchanged if i_L were replaced by $i_L + j$ for any fixed j, which implies that $\langle \eta_a(j) \rangle_{\mu_{x,c}} = \rho_a(x, c)$ for any j, i.e., the densities under $\mu_{x,c}$ are translation invariant. Equation (5.4) may be viewed as a refined version of (1.3), in which the ambiguity in the $L \to \infty$ limit there has been removed.

Noting that in the $L \to \infty$ limit the generator of the dynamics in the neighborhood of i_L does not involve any boundary terms or any constraints on the densities of the three species beyond $\sum_a \eta_a(j) = 1$, one verifies easily [10] that $\mu_{x,c}$ must be invariant for the infinite-volume two species TASEP dynamics. It then follows that $\mu_{x,c}$ must be a convex combination of the extremal invariant measures for the infinite volume two species TASEP. These measures have been classified in [16]: there is (i) a family of translation invariant measures $\nu^{\lambda_0,\lambda_1}$, defined for $\lambda_0, \lambda_1 \geq 0$, $\lambda_0 + \lambda_1 \leq 1$, in which holes, first class particles, and second class particles have densities λ_0, λ_1, and $1 - \lambda_0 - \lambda_1$, respectively, and (ii) a family of non-translation-invariant "blocking" measures $\hat{\nu}^{m,n}$, where $m, n \in \mathbb{Z} \cup \{-\infty, \infty\}$, $m \leq n$, and m, n are not both infinite: $\hat{\nu}^{m,n}$ is a unit point mass on the configuration $\tau^{m,n}$ given by

$$\tau_i^{m,n} = \begin{cases} 0, & \text{if } i < m, \\ 2, & \text{if } m \leq i < n, \\ 1, & \text{if } n \leq i. \end{cases} \tag{5.5}$$

However, the translation invariance of the densities implies that none of the blocking measures can be present in the superposition giving $\mu_{x,c}$.

Thus there exists a probability measure $\kappa_{x,c}(d\lambda_0, d\lambda_1)$ (which depends also on α, β, and γ) that specifies the weights of the different translation invariant extremal measures which enter into the superposition:

$$\mu_{x,c} = \int_{\lambda_0,\lambda_1 \geq 0, \ \lambda_0 + \lambda_1 \leq 1} \kappa_{x,c}(d\lambda_0, d\lambda_1) \nu^{\lambda_0,\lambda_1}. \tag{5.6}$$

We will see that: (i) for most values of the parameters, $\kappa_{x,c}$ is a point mass, so that $\mu_{x,c}$ is one of the measures $\nu^{\lambda_0,\lambda_1}$, (ii) in some cases, in which x may lie to the left of, within, or to the right of the fat shock discussed in the introduction, $\mu_{x,c}$ is a superposition of the two or three measures corresponding to these possibilities, and (iii) no more complicated superposition can occur. Note that, as a consequence, the current J_1 is the same for all elements of the superposition (and the same holds for J_0 and for $J_2 = 0$). Here is a first result in this direction.

Theorem 5.1 *If $\mu_{x,c}$ is defined by* (5.2) *and the current and densities at x are related by* $J_1 = \rho_0(x,c)(1 - \rho_0(x,c)) = \rho_1(x,c)(1 - \rho_1(x,c))$, *then* $\mu_{x,c} = \nu^{\rho_0(x,c),\rho_1(x,c)}$.

The condition in the theorem that $\rho_0(x,c)(1 - \rho_0(x,c)) = \rho_1(x,c)(1 - \rho_1(x,c))$ corresponds to the zero current of second class particles and leads to the alternatives of (1.6). We see from Table 1 that this theorem determines $\mu_{x,c}$ completely for most but not all values of α, β, γ, x, and c and that the results are consistent with the intuitive picture sketched in the introduction. In summary:

Remark 5.2 It follows from Theorem 5.1 that:
(a) In region I the local state $\mu_{x,c}$ is ν^{α_c,α_c}; in particular, it is independent of x and c.
(b) In region II the local state $\mu_{x,c}$ is $\nu^{1-\alpha,\alpha}$ for $x < x_0$ and $\nu^{\alpha,\alpha}$ for $x > x_0$, i.e., respectively outside and inside the fat shock. Region III is similar: the local state is $\nu^{\beta,\beta}$ for $x < x_1$ and $\nu^{\beta,1-\beta}$ for $x > x_1$.
Other cases are not determined by the theorem:
(c) The local state is not determined by Theorem 5.1 in the interior of the regions where either type 2 or type 0 particles have a linear profile, that is, on the II/III boundary (where the fat shock is not pinned to one or the other end of the system) with $0 < x < x_0$ or $1 > x > x_1$. See Fig. 2(c, d).
(d) The local state is not determined by Theorem 5.1 (i) at $x = x_0$ in region II and on the I/II boundary, where $x_0 = 0$; (ii) at $x = x_1$ in region III and on the I/III boundary, where $x_1 = 1$, or (iii) at $x = x_0 = 0$ and $x = x_1 = 1$ at the triple point. All of these points are edges of the (pinned) fat shock; see Fig. 2(a, b, e).
We will determine $\mu_{x,c}$ in cases (c) and (d) below.

Proof of Theorem 5.1 Using (5.6) together with the relations $\langle \eta_1(i) \rangle_{\nu^{\lambda_0,\lambda_1}} = \lambda_1$, $\langle \eta_1(i)(1 - \eta_1(i+1)) \rangle_{\nu^{\lambda_0,\lambda_1}} = \lambda_1(1 - \lambda_1)$ (which hold for all i because the marginal of $\nu^{\lambda_0,\lambda_1}$ on configurations of first class particles is a product measure), we find that

$$\rho_1(x,c) = \langle \eta_1(i) \rangle_{\mu_{x,c}} = \int \lambda_1 \kappa_{x,c}(d\lambda_0, d\lambda_1) = \langle \lambda_1 \rangle_{\kappa_{x,c}}, \tag{5.7}$$

and

$$J_1 = \langle \eta_1(i)(1 - \eta_1(i+1)) \rangle_{\mu_{x,c}}$$
$$= \int \lambda_1(1 - \lambda_1) \kappa_{x,c}(d\lambda_0, d\lambda_1) = \langle \lambda_1(1 - \lambda_1) \rangle_{\kappa_{x,c}}. \tag{5.8}$$

From $J_1 = \rho_1(x,c)(1 - \rho_1(x,c))$, then, we see that $\langle \lambda_1^2 \rangle_{\kappa_{x,c}} = \langle \lambda_1 \rangle_{\kappa_{x,c}}^2$, so that $\lambda_1 = \langle \lambda_1 \rangle_{\kappa_{x,c}} = \rho_1(x,c)$ almost surely with respect to $\kappa_{x,c}$. Similarly, $\lambda_0 = \rho_0(x,c)$ almost surely with respect to $\kappa_{x,c}$, so that $\mu_{x,c} = \nu^{\rho_0(x,c),\rho_1(x,c)}$. □

We now turn to the determination of the local measure $\mu_{x,c}$ at a point x where the densities are varying linearly or are discontinuous—case (c) or (d) of Remark 5.2. Recall that in Sect. 4 we have determined the probability $\theta_{L,n}^{\alpha,\beta}(j,k,l)$ that $Q_1 = j+1$ and $Q_n = j+k+2$, where $j+k+l = L-2$ and Q_1 and Q_n are the position of the first and last second class particles. Now let $\mu_{L,n,j,k,l}^{\alpha,\beta}$ denote the measure $\mu_{L,n}^{\alpha,\beta}$ conditioned on $Q_1 = j+1$, $Q_n = j+k+2$. The key observation we will use follows from Remark 2.1, specifically, from (2.9) or a simple generalization thereof: if G is a function on $Y_{L,n}$ which depends on the τ_i only for

$m_0 \leq i \leq m_1$, then

$$\langle G \rangle_{\mu_{L,n,j,k,l}^{\alpha,\beta}} = \begin{cases} \langle G \rangle_{\mu_{j,0}^{\alpha,1}}, & \text{if } m_1 \leq j, \\ \langle T^{-j-1} G \rangle_{\mu_{k,n-2}^{1,1}}, & \text{if } j+2 \leq m_0, m_1 \leq i+k+1, \\ \langle T^{-j-k-2} G \rangle_{\mu_{l,0}^{1,\beta}}, & \text{if } j+k+3 \leq m_0. \end{cases} \tag{5.9}$$

Now let (r_L) be a sequence of integers with $r_L \to \infty$ and $r_L/\sqrt{L} \to 0$. For any function F on Y depending only on finitely many spins we may write

$$\langle F \rangle_{\mu_{x,c}} = \lim_{L \to \infty} \sum_{j+k+l=L-2} \theta_{L,n_L}^{\alpha,\beta}(j,k,l) \langle T^{i_L} F \rangle_{\mu_{L,n_L,j,k,l}^{\alpha,\beta}}$$

$$= \lim_{L \to \infty} \left[\Xi_L^{(1)} \langle F \rangle_{\mu_L^{(1)}} + \Xi_L^{(2)} \langle F \rangle_{\mu_L^{(2)}} + \Xi_L^{(3)} \langle F \rangle_{\mu_L^{(3)}} + \text{remainder} \right]. \tag{5.10}$$

Here $\mu_L^{(p)}$ is for $p = 1, 2, 3$ the probability measure defined by

$$\mu_L^{(p)} = \Xi_L^{(p)-1} \sum_{j,k,l}^{(p)} \theta_{L,n_L}^{\alpha,\beta}(j,k,l) T^{-i_L} \mu_{L,n_L,j,k,l}^{\alpha,\beta}, \tag{5.11}$$

where $\sum_{j,k,l}^{(1)}$ ranges over values satisfying $j > i_L + r_L$, $\sum_{j,k,l}^{(2)}$ over $j < i_L - r_L$ and $k > i_L + r_L$, $\sum_{j,k,l}^{(3)}$ over $k < i_L - r_L$, and

$$\Xi_L^{(p)} = \sum_{j,k,l}^{(p)} \theta_{L,n_L}^{\alpha,\beta}(j,k,l) = \begin{cases} \mu_{n,L}^{\alpha,\beta}(Q_1 > i_L + r_L), & \text{if } p = 1, \\ \mu_{n,L}^{\alpha,\beta}(Q_1 < i_L - r_L, Q_n > i_L + r_L), & \text{if } p = 2, \\ \mu_{n,L}^{\alpha,\beta}(Q_n < I_L - r_L), & \text{if } p = 3. \end{cases} \tag{5.12}$$

The remainder in (5.11) contains those terms from the full sum over i and k which are omitted from $\sum^{(1)}$, $\sum^{(2)}$, and $\sum^{(3)}$. We have suppressed the dependence of these various entities on α, β, γ, x, and c.

We now take the $L \to \infty$ limit in (5.10). It follows from Remark 4.1 and the fact that r_L grows more slowly than $\sqrt{L}$ that the remainder vanishes in this limit. The $\Xi_L^{(p)}$ are expressed as probabilities in (5.12) and their limiting values $\Xi_{x,c}^{(p)} = \lim_{L \to \infty} \Xi_L^{(p)}$ may be determined from Remark 4.1; these limits will be discussed on a case by case basis below.

Finally, to study $\lim_{L \to \infty} \mu_L^{(p)}$ we observe that for sufficiently large L (if F depends on τ_i only for $|i| \leq m$ then $r_L > m$ suffices) we have by (5.9) that

$$\langle F \rangle_{\mu_L^{(p)}} = \begin{cases} \Xi_L^{(1)-1} \sum_{j,k,l}^{(1)} \theta_{L,n_L}^{\alpha,\beta}(j,k,l) \langle T^{i_L} F \rangle_{\mu_{j,0}^{\alpha,1}}, & \text{if } p = 1, \\ \Xi_L^{(2)-1} \sum_{j,k,l}^{(2)} \theta_{L,n_L}^{\alpha,\beta}(j,k,l) \langle T^{i_L-j-1} F \rangle_{\mu_{k,n-2}^{1,1}}, & \text{if } p = 2, \\ \Xi_L^{(3)-1} \sum_{j,k,l}^{(3)} \theta_{L,n_L}^{\alpha,\beta}(j,k,l) \langle T^{i_L-j-k-2} F \rangle_{\mu_{l,0}^{1,\beta}}, & \text{if } p = 3. \end{cases} \tag{5.13}$$

The limits $\lim_{L \to \infty} \mu_L^{(p)}$ for $p = 1, 2, 3$ are all treated similarly; let us discuss the case $p = 2$ in detail. Equation (5.13) displays $\mu_L^{(2)}$ as a convex combination of the probability measures $T^{-(i_L-j-1)} \mu_{k,n-2}^{1,1}$. Each of these measures, for large L, is by Remark 5.2(a) approximately equal to $\nu^{\alpha \wedge \beta, \alpha \wedge \beta}$ (recall that $\alpha \wedge \beta = \min\{\alpha, \beta\}$), since the critical value α_c^* of α for a

system with $k \simeq wL = \gamma L/(1 - 2\alpha \wedge \beta)$ sites and $n \simeq \gamma L$ second class particles—and thus an effective value $\gamma^* = n/k = 1 - 2\alpha \wedge \beta$ of γ—is $(1 - \gamma^*)/2 = \alpha \wedge \beta$. The same should be true of $\mu_L^{(2)}$. The corresponding evaluations for $p = 1, 3$ come from the results of [10] for the local measures in the one species open system. We conclude that

$$\lim_{L \to \infty} \mu_L^{(p)} = \begin{cases} \nu^{1-\alpha \wedge (1/2), \alpha \wedge (1/2)}, & \text{if } p = 1, \\ \nu^{\alpha \wedge \beta, \alpha \wedge \beta}, & \text{if } p = 2, \\ \nu^{\beta \wedge (1/2), 1-\beta \wedge (1/2)}, & \text{if } p = 3, \end{cases} \tag{5.14}$$

and so

$$\mu_{x,c} = \Xi_{x,c}^{(1)} \nu^{1-\alpha \wedge (1/2), \alpha \wedge (1/2)} + \Xi_{x,c}^{(2)} \nu^{\alpha \wedge \beta, \alpha \wedge \beta} + \Xi_{x,c}^{(3)} \nu^{\beta \wedge (1/2), 1-\beta \wedge (1/2)}. \tag{5.15}$$

Remark 5.3 The heuristic argument for (5.14) given above could be made precise by justifying the implicit exchange of limits; we sketch instead an alternate proof, again for $p = 2$. We know that the limiting current for the measures $\mu_{k,n-2}^{1,1}$, as L and hence $k \simeq wL$ goes to infinity, is $\alpha(1 - \alpha)$, and the limiting densities of holes, particles, and second class particles are α, α, and $1 - 2\alpha$, respectively. One can in fact show further that these limits are obtained with error which goes to zero uniformly at sites i satisfying $r_L \leq i \leq k - r_L$; from this, it follows that the measures $\mu_L^{(2)}$ have the same limiting current and densities. Then an argument as in the proof of Theorem 5.1 establishes (5.14).

To complete our discussion of the local states in the bulk we must supplement (5.15) with a determination of the weights $\Xi_{x,c}^{(p)} \equiv \lim_{L \to \infty} \Xi_L^{(p)}$ for cases (c) and (d) of Remark 5.2. The cases in the next remark are parallel to those of Remark 4.1.

Remark 5.4 (a) On the boundary of regions II and III (the shock line, case (c) of Remark 5.2) we find from Remark 4.1(a) and (5.13) that

$$\Xi_{x,c}^{(p)} = \frac{1}{1 - w} \times \begin{cases} (1 - w - x)_+, & \text{if } p = 1, \\ 1 - w - (1 - w - x)_+ - (x - w)_+, & \text{if } p = 2, \\ (x - w)_+, & \text{if } p = 3. \end{cases} \tag{5.16}$$

Here $u_+ = u$ if $u \geq 0$ and $u_+ = 0$ if $u < 0$. Note that these coefficients, and hence the local measure $\mu_{x,c}$ given by (5.15), are independent of c.

(b) In region II ($\alpha < \alpha_c, \alpha < \beta$), at the fixed shock x_0, the $\Xi_{x,c}^{(p)}$ do depend on c:

$$\Xi_{x_0,c}^{(1)} = 1 - \Phi\left(c\sqrt{A(\alpha)}\right), \qquad \Xi_{x_0,c}^{(2)} = \Phi\left(c\sqrt{A(\alpha)}\right), \qquad \Xi_{x_0,c}^{(3)} = 0. \tag{5.17}$$

Here Φ is the error function defined by

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\tau^2/2} d\tau. \tag{5.18}$$

The analysis in region III is similar.

(c) On the boundary of regions I and II ($\alpha_c = \alpha < \beta$), where $x_0 = 0$, we find that for $c > 0$,

$$\Xi_{0,c}^{(1)} = 2 - 2\Phi\left(c\sqrt{A(\alpha)}\right), \qquad \Xi_{0,c}^{(2)} = 2\Phi\left(c\sqrt{A(\alpha)}\right) - 1, \qquad \Xi_{0,c}^{(3)} = 0. \tag{5.19}$$

The analysis on the I/III boundary is similar: for $c < 0$,

$$\Xi_{1,c}^{(1)} = 0, \qquad \Xi_{1,c}^{(2)} = 1 - 2\Phi\left(c\sqrt{A(\alpha)}\right), \qquad \Xi_{1,c}^{(3)} = 2\Phi\left(c\sqrt{A(\alpha)}\right). \qquad (5.20)$$

(d) At the triple point ($\alpha_c = \alpha = \beta$), where $x_0 = 0$ and $x_1 = 1$, (5.19) and (5.20) again hold (with $c > 0$ and $c < 0$ respectively).

We finally note that the results of this section yield density profiles as well as the finite volume corrections to these profiles at the fixed shocks (see Remark 1.1(a)), since $\rho_a(x, c)$ may be calculated from (5.4) and (5.15). For example, on the II/III border (shock line) we find in this way that

$$\rho_0(x) = \Xi_{x,c}^{(1)}(1 - \alpha) + (\Xi_{x,c}^{(2)} + \Xi_{x,c}^{(3)})\alpha, \qquad \rho_1(x) = (\Xi_{x,c}^{(1)} + \Xi_{x,c}^{(2)})\alpha + \Xi_{x,c}^{(3)}(1 - \alpha), \quad (5.21)$$

with the weights $\Xi^{(p)}$ given by (5.16); it is easy to see that (5.21) reproduces (1.5). Here we have used the notation $\rho_a(x)$ of (1.3), rather than writing $\rho_a(x, c)$ as in (5.4), since the densities do not depend on c. In region II we have, from (5.15) and (5.16), that

$$\rho_0(x_0, c) = 1 - \alpha - \Phi\left(c\sqrt{A(\alpha)}\right)(1 - 2\alpha). \qquad (5.22)$$

Similar results hold in region III at the point x_1.

6 Local States in the Infinite Volume Limit at the Boundaries

In this section we study limiting measures $\lim_{L\to\infty} T^{-i_L}\mu_{L,n_L}^{\alpha,\beta}$ as in (5.2), still taking $n_L = \lfloor \gamma L \rfloor$ but now assuming that either i_L or $L - i_L$ is fixed. Without loss of generality we can assume that $i_L = 1$ or $i_L = L$ (the measure as seen from site j or site $L - j$ can be recovered from these limits) and thus define

$$\mu_{\text{left}} \equiv \lim_{L\to\infty} T^{-1}\mu_{L,\lfloor\gamma L\rfloor}^{\alpha,\beta}, \qquad \mu_{\text{right}} \equiv \lim_{L\to\infty} T^{-L}\mu_{L,\lfloor\gamma L\rfloor}^{\alpha,\beta}. \qquad (6.1)$$

Note that μ_{left} (respectively μ_{right}) does not coincide with any of the measures $\mu_{0,c}$, $c > 0$, (respectively $\mu_{1,c}$, $c < 0$,) studied in Sect. 5. The densities under μ_{left} and μ_{right} were studied in [2]; for example, $\langle \eta_a(j) \rangle_{\mu_{\text{left}}}$ is denoted $\rho_{\text{left},j}^a$ in [2].

By the particle hole symmetry it suffices to consider μ_{left}, which is a measure on the semi-infinite configuration space $\{0, 1, 2\}^{\{0,1,2,3,...\}}$. In general we do not have a proof that the limit defining μ_{left} exists (except along subsequences), although we expect this to be true; see also the comment below Theorem 6.1. The next result, however, gives a somewhat surprising property which any (subsequence) limit must satisfy; to simplify notation, we will speak as if the limit itself exists.

Theorem 6.1 *The distribution of first class particles under the measure μ_{left} is Bernoulli with a constant density ρ, where ρ is given by*

$$\rho = \begin{cases} \alpha_c, & \text{in region I,} \\ \alpha, & \text{in region II,} \\ \beta, & \text{in region III.} \end{cases} \qquad (6.2)$$

Proof By Theorem 3.2, we know that the (marginal) distribution of the variables $\eta_1(i)$ under μ_{left} is exchangeable, so that by de Finetti's theorem [21] this distribution is a superposition of Bernoulli distributions. From [2] we know that for any $i \geq 0$, $\rho \equiv \langle \eta_1(i) \rangle_{\mu_{\text{left}}} = \lim_{L \to \infty} \langle \eta_1(i) \rangle_{\mu_{L,\lfloor \gamma L \rfloor}^{\alpha,\beta}}$ is given by (6.2) and that $\lim_{L \to \infty} \langle \eta_1(i)(1 - \eta_1(i+1)) \rangle_{\mu_{L,\lfloor \gamma L \rfloor}^{\alpha,\beta}} = J_1$ (see (1.1)). In each region of the phase plane these limits satisfy the relation $J = \rho(1 - \rho)$. Then from an argument as in the proof of Theorem 5.1 it follows that the $\eta_1(i)$ distribution is the product measure ν^ρ, where ρ is given by (6.2). $\qquad\square$

Note that in region II the density of second class particles any finite distance from the left boundary goes to zero as $L \to \infty$ [2], so that knowing that the distribution of particles is Bernoulli completely determines any limiting measure to be a Bernoulli measure on particles and holes only. It follows that the limiting measure exists without passing to subsequences.

We discuss briefly one aspect of the measure μ_{left} in the limit $\gamma \to 0$ (note that we are taking this limit *after* the $L \to \infty$ limit). Consider first region I; from Remark 4.1(e) we see that the position Q_1 of the first second class particle in the system is distributed according to $p^{1/4,\alpha}(q_1)$; this is a normalizable distribution which decreases as $q_1^{-3/2}$ for large q_1, so that there remains a second class particle in the system, but $\langle Q_1 \rangle_{\mu_{\text{left}}} = \infty$. In fact more is true; by a calculation similar to that of Remark 4.1 one can show that all $Q_j - Q_{j-1}$, $j = 2, 3, \ldots$, have this same distribution (see also the discussion of the pressure ensemble in Sect. 7) so that there remain an infinite number of second class particles in the system under μ_{left}. The same is true in Region III, but there by Remark 4.1(b) Q_1 is distributed according to $p^{\beta(1-\beta),\alpha}$, so that $\langle Q_1 \rangle_{\mu_{\text{left}}} < \infty$; the distribution of the $Q_j - Q_{j-1}$, $j = 2, 3, \ldots$, is the same as in Region I.

Remark 6.2 One may compare Theorem 6.1 with the result in [9] for the infinite volume limit of a two-component TASEP system on a ring: that the distribution of first class particles to the right of a second class particle, and that of holes to the left of such a particle, is Bernoulli. The two results are closely related, because if we set $\alpha = \beta = 1$ in the open system then the matrix element $\langle W_1 | X_{\tau_1} \cdots X_{\tau_L} | V_1 \rangle$ giving the weight of the configuration $\tau_1, \ldots, \tau_L$ is [9] exactly the weight of the configuration $2, \tau_1, \ldots, \tau_L$ on a ring. Because the numbers of first class particles and of holes on the ring is fixed, and these numbers fluctuate in the open system, this does not establish an exact equivalence of the $\alpha = \beta = 1$ case of Theorem 6.1 to the result of [9]; nevertheless, it is clear that the former is in some sense a generalization of the latter to values of α and β other than 1. But the result of [9] is in another sense more general than Theorem 6.1, since the infinite volume limit of the open system has zero current of second class particles, but this is not true for the system of [9].

7 The Pressure Ensemble for Second Class Particles

We here consider the steady state distribution of the second class particles only, so that one may think of identifying the first class particles and holes as a new type of hole. For d a positive integer we define

$$\phi_\alpha(d) = -\log(4^{-d} Z^{\alpha,1}(d-1,0)) = -\log(4^{-d} \langle W_1 | (X_0 + X_1)^{d-1} | V_1 \rangle). \qquad (7.1)$$

It follows from the (α, β) symmetry of $Z^{\alpha,\beta}(L,n)$ that $\phi_\beta(d)$ is also equal to $-\log(4^{-d} Z^{1,\beta}(d-1,0))$. Using (2.8) we find that the probability that the n second class

particles in the systems are located at sites $q_1 < q_2 < \cdots < q_n$ is

$$
\mu_{L,n}^{\alpha,\beta}(Q_1 = q_1, \ldots, Q_n = q_n)
$$
$$
= (4^{-L} Z^{\alpha,\beta}(L,n))^{-1} e^{-\phi_\alpha(q_1) - \sum_{i=2}^n \phi(q_i - q_{i-1}) - \phi_\beta(L - q_n)}, \qquad (7.2)
$$

where we have denoted $\phi_1(d)$ by $\phi(d)$. The motivation for the factors 4^{-d} in (7.1) will be discussed below; with this normalization $\phi(d) \sim (3/2) \log d$ for $d \to \infty$ [9, 17].

We note that (7.2) has precisely the form of the canonical distribution for a system in a domain of length L with particles interacting with their nearest neighbor only via a pair potential $\phi(d)$. (Such an interaction is rather unphysical; one may think of any intervening particle as screening the interaction of particles that it separates.) There is also a potential $\phi_\alpha(d)$ $(\phi_\beta(d))$ representing the interaction of the first (last) particle with the left (right) boundary.

The TASEP dynamics for the full system gives rise in a natural way to a dynamics on the system of the second class particles which satisfies detailed balance with respect to this equilibrium measure. In the state in which the second class particles are at $q_1, \ldots, q_n$ the ith second class particle moves to site $q_i + 1$ at rate 1 whenever that site is empty (in the original sense), an event which by a simple generalization of (2.9) occurs in the NESS with probability

$$
\frac{\langle W_1 | X_0 (X_0 + X_1)^{q_{i+1} - q_i - 2} | V_1 \rangle}{Z^{1,1}(q_{i+1} - q_i - 1, 0)} = \frac{e^{-\phi(q_{i+1} - q_i - 1)}}{e^{-\phi(q_{i+1} - q_i)}}, \quad \text{if } i < n, \qquad (7.3)
$$

and with probability

$$
\frac{\langle W_1 | X_0 (X_0 + X_1)^{L - q_i - 1} | V_\beta \rangle}{Z^{1,\beta}(L - q_i, 0)} = \frac{e^{-\phi_\beta(L - q_i - 1)}}{e^{-\phi_\beta(L - q_i)}}, \quad \text{if } i = n. \qquad (7.4)
$$

One finds similarly that the probability that the site $q_i - 1$ is occupied by a first class particle is

$$
\frac{e^{-\phi(q_i - q_{i-1} - 1)}}{e^{-\phi(q_i - q_{i-1})}}, \quad \text{if } i > 1, \qquad \frac{e^{-\phi_\alpha(q_1 - 1)}}{e^{-\phi_\alpha(q_i)}}, \quad \text{if } i = 1. \qquad (7.5)
$$

The dynamics in which $q_i \to q_i + 1$ when $q_{i+1} - q_i \geq 2$, with rate given by (7.3), and $q_i \to q_i - 1$ when $q_i - q_{i-1} \geq 2$, with rate given by (7.5), is easily seen to satisfy detailed balance with respect to the measure (7.2).

To obtain the properties of the system described by (7.2) in the thermodynamic limit, $L \to \infty, n/L \to \gamma$, it is most convenient to consider the *pressure* or *isobaric ensemble* $\pi_{p,n}^{\alpha,\beta}$ [23, 24], in which instead of keeping the volume L of the system fixed we imagine that the right wall is in contact with a reservoir of pressure p. The value of p can be chosen so as to make the average volume equal to L, as discussed below. More precisely, we let the position of the right boundary, which we denote q_{n+1}, fluctuate, and add a term involving the pressure p to the measure. This yields the probability distribution in the pressure ensemble:

$$
\pi_{p,n}^{\alpha,\beta}(q_1, \ldots, q_n, q_{n+1})
$$
$$
= \mathcal{Z}^{\alpha,\beta}(p,n)^{-1} \exp\left(-\phi_\alpha(q_1) - \sum_{i=2}^n \phi(q_i - q_{i-1}) - \phi_\beta(L - q_n) - p q_{n+1} \right). \quad (7.6)
$$

The partition function has the form $\mathcal{Z}^{\alpha,\beta}(p,n) = \mathcal{Z}_1(\alpha, p)\mathcal{Z}_2(p)^n\mathcal{Z}_1(\beta, p)$, where $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are readily found, for $z = \sqrt{1 - e^{-p}}$ satisfying

$$1 \geq z \geq \max\{0, 1 - 2\alpha, 1 - 2\beta\}, \tag{7.7}$$

to be given by

$$\mathcal{Z}_1(\alpha, p) = \frac{\alpha(1 - z)}{z + 2\alpha - 1}, \qquad \mathcal{Z}_2(p) = \frac{1 - z}{1 + z}. \tag{7.8}$$

Thus (7.6) becomes

$$\pi_{p,n}^{\alpha,\beta}(q_1, \ldots, q_n, q_{n+1})$$
$$= \mathcal{Z}_1(\alpha, p)^{-1}e^{-\phi_\alpha(q_1)-pq_1}\left[\prod_{i=2}^n \mathcal{Z}_2(p)^{-1}e^{-\phi_\alpha(q_i-q_{i-1})-p(q_i-q_{i-1})}\right]$$
$$\times \mathcal{Z}_1(\beta, p)^{-1}e^{-\phi_\beta(q_{n+1}-q_n)-p(q_{n+1}-q_n)}. \tag{7.9}$$

The convenient factorization property of the probability $\pi_{p,n}^{\alpha,\beta}(q_1, \ldots, q_n, q_{n+1})$ displayed in (7.9), which implies that the variables q_1 and $q_j - q_{j-1}$, $j = 2, \ldots, n + 1$, are independent, has prompted the use of the pressure ensemble for equilibrium systems, without any reference to dynamics. The requirement that particles only interact with their first neighbors is usually imposed artificially (see, however, [25]). In our model this condition arises naturally from the dynamics. Note that $q_n - q_1$, the width of the fat shock, is thus represented as a sum of independent random variables; this is consistent with its Gaussian distribution in regions I and II of the fixed volume ensemble (see Remark 4.1).

One easily checks that (writing now $\pi_{p,n}^{\alpha,\beta} = \pi$)

$$\langle q_1 \rangle_\pi = -\frac{d}{dp}\log\mathcal{Z}_1(\alpha) = \frac{\alpha(1 + z)}{z(z + 2\alpha - 1)},$$
$$\langle q_j - q_{j-1}\rangle_\pi = -\frac{d}{dp}\log\mathcal{Z}_2 = \frac{1}{z}, \quad j = 2, \ldots, n, \tag{7.10}$$
$$\langle q_{n+1} - q_n\rangle_\pi = -\frac{d}{dp}\log\mathcal{Z}_1(\beta) = \frac{\beta(1 + z)}{z(z + 2\beta - 1)}.$$

Note that when z approaches its lower bound, which is 0 if $\alpha, \beta \geq 1/2$ and $\max\{1 - 2\alpha, 1 - 2\beta\}$ otherwise, the average size $\langle q_{n+1}\rangle_\pi$ of the system goes to infinity for every $n \geq 1$; there is simply not enough pressure to confine the system. To agree with standard definitions we have defined the potentials ϕ_α in (7.1) so the size of the system in the absence of boundary terms, that is, $\langle q_n - q_1\rangle_\pi$, goes to infinity when $p \to 0$ ($z \to 0$).

To find the appropriate pressure corresponding to the canonical ensemble with $L = n/\gamma$ studied above we must set the expected system length

$$\langle q_{n+1}\rangle_\pi = \langle q_1\rangle_\pi + \sum_{j=2}^n \langle q_j - q_{j-1}\rangle_\pi + \langle q_{n+1} - q_n\rangle_\pi = -\frac{d}{dp}\log\mathcal{Z}^{\alpha,\beta}(p,n), \tag{7.11}$$

equal to L and solve for p (or z), subject to the restrictions (7.7). With (7.10) the equation to be solved becomes

$$\frac{\alpha(1 + z)}{z(z + 2\alpha - 1)} + \frac{n}{z} + \frac{\beta(1 + z)}{z(z + 2\beta - 1)} = \frac{n}{\gamma}. \tag{7.12}$$

We will discuss the solution of this equation in various regions of the phase plane; it is useful to bear in mind that each of the three terms on the left hand side increases as z decreases from 1 to its lower limit $\max\{0, 1 - 2\alpha, 1 - 2\beta\}$.

Consider first region I, where $\alpha, \beta > (1 - \gamma)/2$. Since for $z = \gamma$ the left hand side of (7.12) is $n/\gamma + O(1)$, where the $O(1)$ term is positive, the solution must be of the form $z = \gamma + O(1/n)$. In the limit $n \to \infty$ we thus have $z = \gamma$ or $p = -\log(1 - \gamma^2)$. In this case $\langle q_1 \rangle_\pi$ and all $\langle q_j - q_{j-1} \rangle_\pi$, $j = 2, \ldots, n + 1$, are of order unity, so the bulk of the system has, in the limit $n \to \infty$, the same structure as that obtained from our NESS when $L \to \infty$ in region I.

In Region II, where $\alpha < \beta$ and $\alpha < (1 - \gamma)/2$, we have from $z > 1 - 2\alpha > 1 - 2\beta$ that the third term in (7.12) is $O(1)$, and from $z > 1 - 2\alpha > \gamma$ that the first term must be $O(n)$, i.e., we must have $z = 1 - 2\alpha + O(1/n)$. In fact we find easily that for large n,

$$z \simeq 1 - 2\alpha + \frac{2\alpha(1 - \alpha)}{1 - 2\alpha - \gamma} \frac{1}{n}. \tag{7.13}$$

Now $\langle q_1 \rangle_\pi$ is of order n but $\langle q_{n+1} - q_1 \rangle_\pi$ is still of order 1; this corresponds to the fat shock being fixed to the right wall, i.e., to what we see in region II. The situation in region III is of course obtained by interchanging α with β and left with right. In the case when $\alpha = \beta < (1 - 2\gamma)/2$ one sees again that $z \simeq 1 - 2\alpha + c/n$ and that $\langle q_1 \rangle_\pi = \langle q_{n+1} - q_n \rangle_\pi$ are both of order n; this simply means that the average position of the fat shock is in the middle.

One may of course analyze the pressure ensemble directly, rather than looking for the correspondence with the open system of fixed length; the key question is how one allows p or equivalently z to vary with the number n of (second class) particles. If z is held fixed (necessarily in the range (7.7)) then the behavior of the system corresponds to that of the open system in region I. If $\alpha < 1/2$ and $\alpha < \beta$, and one takes $z = 1 - 2\alpha + c/n$ then the behavior is as in region II; similarly if $\beta < 1/2$ and $\beta < \alpha$ one obtains region III behavior by taking $z = 1 - 2\beta + c/n$, and if $\alpha = \beta < 1/2$ such a z value gives behavior corresponding to the II/III boundary. A detailed analysis of the ensemble, for example of the shape of the profiles of the fat shock, would essentially repeat the analysis in the fixed L, i.e., fixed overall density γ, ensemble studied earlier, and we will not take up these questions again here.

Note that for almost all permissible values of the pressure our system is in region I; only by fine tuning the pressure to change with n in a range of order $1/n$ do we get configurations as in regions II or III. This is reminiscent of what happens when one goes from a fixed magnetization to a fixed external magnetic field h in the Ising model at low temperatures, in dimension two or higher. The whole coexistence region, corresponding to the average magnetization being smaller than the spontaneous magnetization, corresponds to the single value $h = 0$.

Other choices of z can lead to regimes different from those considered in the present work. For example, if we again suppose that $\alpha = \beta < 1/2$, but now take z closer than order $1/n$ to $1 - 2\alpha$—to be specific, say $z = 1 - 2\alpha + c/n^2$—then $\langle q_{n+1} \rangle_\pi$ is of order n^2 and hence the density of second class particles is zero.

8 Concluding Remarks

1. As noted already in several places above, the local properties of our system away from the boundaries approach, as $L \to \infty$, those of the states of the two species TASEP on the lattice $\mathbb{Z}$. Because of this it will be useful to describe here some known properties of the

(extremal, translation invariant) NESS's of that system, i.e., of the states ν^{ρ_0,ρ_1} introduced in Sect. 5. These states differ from those of other models for which the NESS of the finite open system can be solved exactly, such as the one species simple exclusion process and the zero range processes [9, 26, 27], in that they are not product measures; this is so despite the fact that their projections (marginals) on the configurations of first class particles alone, or on the configurations of holes alone, are in fact Bernoulli. The states ν^{ρ_0,ρ_1} may be obtained [9] as the $N \to \infty$ limits of states of a two component TASEP on a ring of N sites, with $N_\alpha = \rho_\alpha N$ particles of type α, $\alpha = 0, 1, 2$, where $\rho_2 = 1 - \rho_0 - \rho_1$; see also [16].

As noted in [9], the structure of the states ν^{ρ_0,ρ_1} is quite intricate, containing several mysterious features which we still do not understand in any direct intuitive way. They are not even Gibbs measures [16], even though all the truncated correlation functions involving a finite number of sites decay exponentially fast. This decay follows from the (mysterious) fact that, conditioned on the presence of a second class particle at site i, the measure factorizes: the left and right sides of i become independent. For the corresponding property for the open system studied in this paper see Remark 2.1. Another (mysterious) fact is that if one conditions on the presence of a second class particle at i then the particles to the right of i, and the holes to the left of i, have this Bernoulli property [9].

Another related property of the states ν^{ρ_0,ρ_1} is that if one conditions on there being a first class particle (respectively a hole) at site i then the measure to the left (respectively right) of i is the same as if there was no conditioning at all, i.e., the same as that described in the first paragraph of this section. (This may be expressed colloquially by saying that if one observes that the fastest horse is in front then one gains no information about the rest.) The property has in fact been established in not only the two species but also the n-species TASEP (see [28], Proposition 6.2), using a representation of the stationary measure based on queuing theory; a direct proof for the two species model may be given using the two properties of second class particles noted in the previous paragraph. We remark that the property of factorization around a second class particle does not extend in a direct way to the n-species model [28].

2. The fact that the measures ν^{ρ_0,ρ_1} are not product measures gives extra structure to the local states $\mu_{x,c}$ discussed in Sect. 5, which are superpositions of such measures. We note here however that as in the case of the one component TASEP, when such a superposition occurs only on the shock line $\alpha = \beta < 1/2$, the translation invariant measures which enter into the superposition (and which correspond to the measures on one side or another of a shock) all have the same current. This can be interpreted as saying that the properly averaged local current does not fluctuate. These averages can be obtained either as long time averages of the stochastic flux across a single bond, or as spatial averages over an interval of length K, with $K \to \infty$ *after* $L \to \infty$. We believe that the convergence of the average total flux across an open system to a deterministic value, as $L \to \infty$, is a general property of the NESS of systems like those discussed here, but do not know how to prove this directly at the present time. It seems reasonable to expect similar behavior in higher dimensions and different settings, e.g., for driven diffusive systems on a torus [29].

3. It follows from the "separating" property of conditioning on the presence of a second class particle at a given site that the distribution under ν^{ρ_0,ρ_1} of the second class particles alone is given by a renewal process [9, 17]. When the current J_2 vanishes, i.e., when $\rho_0 = \rho_1 = (1 - \rho_2)/2$, then (as noted in Sect. 7) the distribution of the distance between nearest neighbor particles in this process has a simple exponential dependence on ρ_2 which can be obtained from a pressure ensemble, with $p = -\log(1 - \rho_2^2)$, as in region I of the open system.

Combining this expression for the pressure as a function of the density with standard thermodynamic relations we can obtain expressions for the chemical potential λ and Helmholtz free energy a in the uniform infinite system of second class particles with density ρ_2:

$$\lambda(\rho_2) = -\log\left(\frac{1-\rho_2}{1+\rho_2}\right),$$

$$a(\rho_2) = (1-\rho_2)\log(1-\rho_2) + (1+\rho_2)\log(1+\rho_2). \tag{8.1}$$

From (8.1) we can obtain the large deviation function for the probability of finding $\tilde{\rho}_2 L$ particles in an interval of L lattice sites [30] for any other density $\tilde{\rho}_2$. The large deviation for first class particles or holes alone is of course given by the properties of the Bernoulli measure. Large deviation properties of the full measure have not, so far as we know, been determined for the two species system.

4. Even knowing fully the properties of the infinite system still leaves open the problem of how fast the correlations in the vicinity of a site $\lfloor xL \rfloor$ approach those in the local measure $\mu_{x,c}$. This may be of particular interest in the case when $\mu_{x,c}$ is a superposition of extremal infinite volume measures ν^{ρ_0,ρ_1} as discussed in Sect. 5. We might expect the L dependence in that case, where typical density profiles differ from average ones, to be different from that where the two coincide. We leave this as an open problem.

5. We now briefly describe two related model systems, containing both first and second class particles on a ring, which are intermediate between those studied in [9] and in this paper.

5.1 Recall the "truck" or "defect particle" model [12, 31–33], a standard two species TASEP system: a single defect particle together with (first class) particles and holes, on a ring of $L + 1$ sites, can exchange with a hole ahead of it (clockwise) at rate α and with a particle behind it at rate β. Let us add to the ring also n (standard) second class particles, which make the same exchanges as does the defect particle but at rate 1 in each case, and which do not exchange at all with the defect particle. To be definite let us say that there are n_1 first class particles and n_0 holes on the ring, with $n + n_1 + n_0 = L$. Then the stationary measure for this system is almost the same as that of our open system: using the matrices X_0, X_1, and X_2 of Sect. 2, and introducing also $X_\delta = |V_\beta\rangle\langle W_\alpha|$, we find that a configuration $\delta, \tau_1, \ldots, \tau_L$, where δ represents the defect particle, has weight:

$$\text{Tr}(X_\delta X_{\tau_1} \cdots X_{\tau_L}) = \langle W_\alpha | X_{\tau_1} \cdots X_{\tau_L} | V_\beta \rangle, \tag{8.2}$$

(compare (2.4)). The difference, of course, is that this is a canonical ensemble and the partition function must be obtained by summing the weights over only those configurations with the proper numbers of all species. This relation between this truck model on a ring of $L + 1$ sites and the two species open system studied in this paper is completely parallel to that between the usual defect particle model and the open one species TASEP. We expect that details of the stationary state could be worked out in parallel to that of the usual defect particle model, but we have not done so.

5.2 In the second model, the ring has N sites labeled by $i \in [-N/2 + 1, N/2]$ and contains $N_1 = \bar{\rho}_1 N$ first class particles, $N_2 = \bar{\rho}_2 N$ second class particles, and $N_0 = N - N_1 - N_2 = \bar{\rho}_0 N$ holes. The particles jump clockwise according to the TASEP rules given in Sect. 1, *except* at one specified semi-permeable bond, say between site 0 and site 1, which prohibits the passage of second class particles. (We can think of this bond as a restriction in a channel).

Unfortunately we do not have an exact solution for this system. To see what happens, however, we note that, as in the open system, the current J_2 of second class particles will

ONE SEMI-PERMEABLE BOND: $N_0 = 450$; $N_1 = 300$; $N_2 = 250$

ONE SEMI-PERMEABLE BOND: $N_0 = 375$; $N_1 = 375$; $N_2 = 250$

Holes: ····· Second class: ······· First class: ——
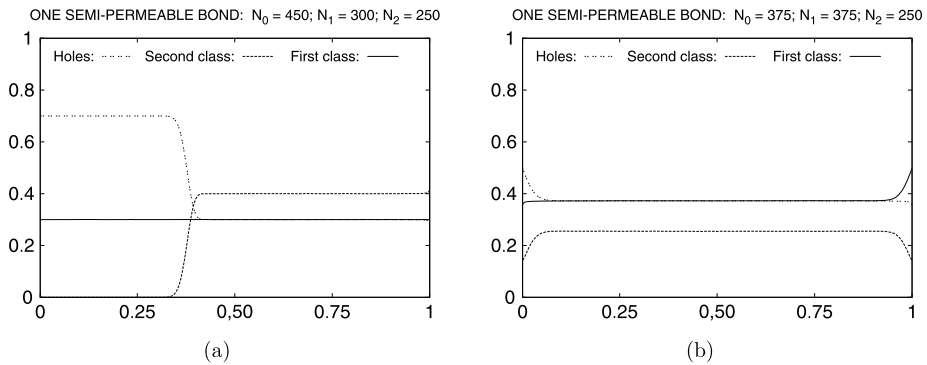
(a)

(b)

Fig. 4 Density profiles in a system with one semi-permeable bond: $L = 1000$

vanish in the stationary state. On the other hand, since we would have $J_2 = \bar{\rho}_2(\bar{\rho}_0 - \bar{\rho}_1)$ if the system were uniform, a uniform state is possible only if $\bar{\rho}_1 = \bar{\rho}_0$. If $\bar{\rho}_1 < \bar{\rho}_0$ then J_2 would be positive in the uniform system and second class particles would drift to the right; the upshot is that there will be a fat shock of width $w = \bar{\rho}_2 N/(1 - 2\bar{\rho}_1)$ containing all second class particles at density $\rho_2 = 1 - 2\rho_1$ pinned to the back of the barrier. If $\bar{\rho}_1 > \bar{\rho}_2$ then the fat shock of width $w = \bar{\rho}_2 N/(1 - 2\bar{\rho}_0)$ will be pinned to the front of the barrier. In the case $\bar{\rho}_0 = \bar{\rho}_1 = (1 - \bar{\rho}_2)/2$ the system will be uniform. See Fig. 4 for some typical profiles in this system; note that N_0, N_1, and N_2 have been chosen so that the bulk densities in figures (a) and (b) here are the same as those in figures (a) and (e) of Fig. 2, but that the boundary effects and finite density shock transition are noticeably different in the two models.

Letting $N \to \infty$ with $\bar{\rho}_1, \bar{\rho}_0$ fixed would yield an infinite system with a barrier at the bond $(0, 1)$. Consider first the case $\bar{\rho}_1 < \bar{\rho}_0$. To the right of the origin there would be no second class particles and a uniform density of first class particles described by a product measure. Far to the left of the barrier the state would be $\nu^{\bar{\rho}_0, \bar{\rho}_0}$, i.e., a uniform state with $\rho_1 = \rho_0 = \bar{\rho}_1$ and $\rho_2 = 1 - 2\bar{\rho}_1$. We do not know, however, the structure of the system just to the left of the barrier. Similar conclusions hold for $\bar{\rho}_1 > \bar{\rho}_0$.

Appendix A: A Particular Representation

A representation of the algebra (2.2)–(2.3) which satisfies (2.6) may be obtained from [7] and [9]:

$$
X_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & . & . \\ 0 & 1 & 1 & 0 & & \\ 0 & 0 & 1 & 1 & & \\ 0 & 0 & 0 & 1 & . & \\ . & & & & . & . \\ . & & & & & . \end{pmatrix}, \qquad X_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & . & . \\ 1 & 1 & 0 & 0 & & \\ 0 & 1 & 1 & 0 & & \\ 0 & 0 & 1 & 1 & & \\ . & & & & . & . \\ . & & & & . & . \end{pmatrix}. \tag{A.1}
$$

$$X_2 = X_1 X_0 - X_0 X_1 = [X_1, X_0] = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & & \\ \cdot & & & & \cdot & \\ \cdot & & & & & \cdot \end{pmatrix}, \qquad (A.2)$$

$$\langle W_\alpha | = \left(1, \left(\frac{1-\alpha}{\alpha} \right), \left(\frac{1-\alpha}{\alpha} \right)^2, \dots \right), \qquad |V_\beta\rangle = \begin{pmatrix} 1 \\ \frac{1-\beta}{\beta} \\ (\frac{1-\beta}{\beta})^2 \\ \vdots \end{pmatrix}. \qquad (A.3)$$

The exponential growth of the components of $\langle W_\alpha |$ and $|V_\beta\rangle$ for certain values of α and β in fact causes no concern here: because we always have $n > 0$, the matrix product needed to calculate the probability of any configuration τ (see (2.4)) will contain at least one factor X_2, and using (2.6) one can see that this implies that the corresponding matrix element is finite.

Appendix B: Asymptotics of the Partition Function

We summarize here the asymptotics of the partition function which are needed in Sect. 5. For the case with no second class particles [7] we need $Z^{\alpha,\beta}$ only when $\alpha = 1$ and/or $\beta = 1$:

$$Z^{\alpha,1}(j,0) = Z^{1,\alpha}(j,0) \sim \begin{cases} \frac{1-2\alpha}{(1-\alpha)^2} \left(\frac{1}{\alpha(1-\alpha)} \right)^j, & \text{if } \alpha < 1/2, \\[2mm] \frac{2}{\sqrt{\pi}} \frac{4^j}{j^{1/2}}, & \text{if } \alpha = 1/2, \\[2mm] \frac{\alpha^2}{\sqrt{\pi}(2\alpha-1)^2} \frac{4^{j+1}}{j^{3/2}}, & \text{if } \alpha > 1/2. \end{cases} \qquad (B.1)$$

The generating function is [12]

$$\sum_{L=1}^{\infty} \lambda^L Z_{L,0}^{\alpha,\beta} = \left(\frac{2\alpha}{2\alpha - 1 + \sqrt{1-4\lambda}} \right) \left(\frac{2\beta}{2\beta - 1 + \sqrt{1-4\lambda}} \right). \qquad (B.2)$$

For the model with second class particles [2]:

- In region I, $(\alpha_c < \alpha, \beta)$

$$Z^{\alpha,\beta}(L,n) = \frac{n\alpha\beta\sqrt{L^2-n^2}}{\sqrt{\pi L}((2\alpha-1)L+n)((2\beta-1)L+n)} \left(\frac{4L^2}{L^2-n^2} \right)^{L+1} \left(\frac{L-n}{L+n} \right)^n; \qquad (B.3)$$

- In region II, $(\alpha < \alpha_c, \alpha < \beta)$

$$Z^{\alpha,\beta}(L,n) = \frac{\alpha\beta(1-2\alpha)}{(\beta-\alpha)} \left(\frac{1}{\alpha(1-\alpha)} \right)^{L+1} \left(\frac{\alpha}{1-\alpha} \right)^n; \qquad (B.4)$$

- On the boundary of regions I and II, $(\alpha_c = \alpha < \beta)$

$$Z^{\alpha,\beta}(L,n) = \frac{\beta n(L-n)}{2L((2\beta-1)L+n)} \left(\frac{4L^2}{L^2-n^2} \right)^{L+1} \left(\frac{L-n}{L+n} \right)^n \qquad (B.5)$$

- On the boundary of regions II and III, $(\alpha = \beta < \alpha_c)$

$$Z^{\alpha,\beta}(L,n) = \frac{(1-2\alpha)((1-2\alpha)L - n)}{(1-\alpha)^2} \left(\frac{1}{\alpha(1-\alpha)}\right)^L \left(\frac{\alpha}{(1-\alpha)}\right)^n \qquad (B.6)$$

- At the triple point, $(\alpha_c = \alpha = \beta)$

$$Z^{\alpha,\beta}(L,n) = \frac{n(L-n)}{2L(L+n)} \sqrt{\frac{L^2 - n^2}{L\pi}} \left(\frac{4L^2}{L^2 - n^2}\right)^{L+1} \left(\frac{L-n}{L+n}\right)^n. \qquad (B.7)$$

Asymptotics in Region III and on the I/III boundary are obtained from those of Region II and the I/II boundary by exchange of α and β.

Appendix C: Finite Volume Corrections to Density Profiles

We consider here again the problem of finding asymptotic values of the density profiles, beginning with a discussion of the method of [2]. The partition function can be expressed as

$$Z^{\alpha,\beta}(L,n) = \frac{\alpha\beta}{\alpha - \beta} \left[R(L,n,\beta) - R(L,n,\alpha)\right], \qquad (C.1)$$

where

$$R(L,n,\alpha) = \sum_{k=0}^{L-n} C_{L-n-k}^{L+n-1} \frac{1}{\alpha^{k+1}}, \qquad (C.2)$$

with C_n^m the Catalan triangle numbers (3.8). An asymptotic analysis of (C.2) then leads, through (C.1) and the formulas (3.13)–(3.14) for the densities, to the density asymptotics. In [2] the asymptotic density at position x was calculated as

$$\lim_{L\to\infty} \langle \eta_a(i_L) \rangle_{\mu_{L,\lfloor\gamma L\rfloor}^{\alpha,\beta}}, \qquad (C.3)$$

with $i_L = \lfloor xL \rfloor$. As observed in Sect. 5, however, if x is the location of the fixed shock in regions II or III, and one considers limits as in (C.3) with $i_L = \lfloor xL \rfloor + c\sqrt{L}$, then the limiting density value depends on c. This c dependence may be calculated by the methods of Sect. 5 (see for example (5.22)); here we sketch briefly an alternate and more direct method which extends the work of [2].

The key step is the computation of the asymptotics of $R(L,n,\alpha)$; it is convenient to introduce $\alpha_c = (L-n)/(2L)$ (see (1.2)). We must determine which terms in (C.2) dominate the sum. If we let $L \to \infty$ at fixed n and α there are three regimes: (i) $\alpha > \alpha_c$, for which the maximum of the summand is attained when k is of order L and the sum can be approximated by a Gaussian integral; (ii) $\alpha < \alpha_c$, in which the maximum is attained when k is of order $-L$ and the sum can be approximated by a geometric series; and (iii) $\alpha = \alpha_c$, for which the maximum occurs when k is of order 1 and the sum can be approximated by half of a Gaussian integral. However, there are intermediate regimes in which the sum is dominated by terms in which k is of order $\pm\sqrt{L}$, and it is these which generate the finite volume density corrections that we seek.

One needs an asymptotic estimate of $R(L, n, \alpha)$ which holds for all large L and n. Such an estimate is $R(L, n, \alpha) \sim \tilde{R}(L, n, \alpha)$, where

$$
\tilde{R}(L, n, \alpha) = \begin{cases}
(1 - 2\alpha)(\frac{1}{\alpha(1-\alpha)})^{L+1}(\frac{\alpha}{1-\alpha})^n \Phi(\frac{L(1-2\alpha)-n}{\sqrt{\alpha(L+n)}}) & \alpha \leq \alpha_c, \\[2mm]
\frac{2nL}{(n-L(1-2\alpha))\sqrt{\pi L(L^2-n^2)}}(\frac{4L^2}{L^2-n^2})^L \\[2mm]
\quad \times (\frac{L-n}{L+n})^n \Psi(\frac{L(1-2\alpha)-n}{\sqrt{\alpha(L+n)}}), & \alpha > \alpha_c;
\end{cases}
\tag{C.4}
$$

here Φ is as in (5.18) and $\Psi(t) = \sqrt{2\pi} e^{t^2/2} |t| \Phi(t)$. The asymptotic estimate $R \sim \tilde{R}$ holds in the sense that for α and the ratio n/L uniformly bounded away from 0 and 1 the quantity $|R/\tilde{R} - 1|$ is small when L is large—more precisely, for any $\epsilon > 0$ there is a constant C_ϵ such that $|R/\tilde{R} - 1| \leq C_\epsilon L^{-1/2-\epsilon}$. We remark that the two forms in (C.4) in fact agree for $\alpha_c < \alpha < \alpha_c + O(1/\sqrt{L})$.

From (C.1) and (C.4) one obtains similarly improved asymptotics for the partition function $Z^{\alpha,\beta}(L, n)$, and the full density asymptotics then follows from the exact formulas of [2] or Theorem 3.3.

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